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Abstract: In this paper standard oligopolies are interpreted in two ways, namely as oligopolies without transferable technologies and as oligopolies with transferable technologies. From a cooperative point of view this leads to two different classes of cooperative games. We show that cooperative oligopoly games without transferable technologies are convex games and that cooperative oligopoly games with transferable technologies are totally balanced, but not necessarily convex.

Key-words: Oligopolies, cooperative games, convexity, total balancedness.

1 Introduction

In this paper we consider a cooperative approach to oligopoly situations. Hereby we will distinguish two different types of oligopolies, namely oligopolies with transferable technologies and oligopolies without transferable technologies. The first type is characterized by the fact that a group of cooperating firms is allowed to produce according to the cheapest technology present in this group, whereas such a transfer of technologies is not possible for the second type of oligopolies. An illustrative example of the first type is a collection of potato farmers. Every farmer faces its specific costs for the production of one ton of potatoes, which strongly depend upon the sowing and irrigation techniques used by the farmer. Cooperating farmers are able to exchange their production techniques, i.e. their knowledge, which is a costless operation. An example of an oligopoly without transferable technologies is a group of fishery companies harvesting some species of fish. The

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costs for the production of one ton of fish is heavily related to the type of ships owned by a firm. In such a situation cooperating firms can not transfer their technologies, since such an operation would imply that some firms have to replace their fleet. Another example of an oligopoly without transferable technologies is a group of oil producing countries, each having their own costs for the production of one barrel of oil. For obvious geographical reasons a country like Norway can not produce oil at the same costs as for example Saudi Arabia, even not in case of cooperation.

Usually, oligopoly situations are modeled by means of non-cooperative games. Aumann (1959) introduced two ways of converting a non-cooperative game into a cooperative one. In the first approach every coalition computes the amount of money which they can guarantee themselves regardless what the players outside the coalition do. The second approach computes for every coalition the minimal amount of money which is such that the players outside the coalition can prevent that the players in the coalition get more. Zhao (1999) showed that for the case of transferable technologies these two approaches lead to the same cooperative game. For special cases of oligopolies, e.g. oligopolies without capacity restrictions, Zhao (1999) provides necessary and sufficient conditions for the convexity of these games.

In this paper we focus on the case of oligopolies without transferable technologies. We show that the resulting cooperative game is a convex game in general. Meinhardt (1999) and Driessen and Meinhardt (2000) obtained this result already for the specific case that all firms are symmetric. Moreover, using the same techniques as in the case of oligopolies without transferable technologies, we are able to show that oligopoly games with transferable technologies are totally balanced in general.

Section 2 contains some preliminaries. In section 3 the cooperative oligopoly games, both with and without transferable technologies, are introduced and a formula for the computation of the coalitional values of these games is provided. Section 4 deals with the properties of cooperative oligopoly games. It is shown that cooperative oligopoly games without transferable technologies are convex games and that cooperative oligopoly games with transferable technologies are totally balanced.

2 Preliminaries

We start this section with some notational conventions which will be used throughout this paper. For $a \in I\!R$ we define $a_+ = \max\{a, 0\}$. If N is a finite (player) set and $(X_i)_{i \in N}$ is a collection of non-empty (strategy) spaces then, for every $S \subseteq N$, $S \neq \emptyset$, the Cartesian product $\Pi_{i \in S} X_i$ is denoted by X_S. Moreover, if $X_i \subseteq I\!R$ for every $i \in N$, and $x = (x_i)_{i \in S} \in X_S$ for some $S \subseteq N$, $S \neq \emptyset$, then the sum $\sum_{i \in T} x_i$ is denoted by $x(T)$ for every $T \subseteq S$. If $(c_i)_{i \in S} \in I\!R^S$ for some $S \subseteq N$, $S \neq \emptyset$, then $\underline{c}_S = \min_{i \in S} c_i$.

Consider a monopolistic producer of some good, whose maximum production capacity is $y > 0$. Suppose that this monopolist faces the linear inverse demand function $p(t) = x - t, t \in [0, y]$, with $x \in I\!R$. So, if the monopolist produces t units of output, he can sell these at a price of $x - t$ per unit of output. Note that the monetary scale is chosen in such a way that an increase in output of one unit causes a decrease in price of one unit. For technical reasons we allow x to be non-positive, although the economical interpretation in this case is meaningless. We assume moreover that the monopolist can produce at zero costs. So, the monopolist faces the following simple maximization problem

$$\begin{array}{ll} \text{maximize} & (x - t) \cdot t \\ \text{such that} & t \in [0, y]. \end{array} \tag{1}$$

We will refer to (1) as *monopolistic optimization problem* (x, y). One easily verifies that the maximum of this problem is $f_y(x)$, where the function f_y is provided in the definition below.

Definition 1 For every $y > 0$ the C^1-function $f_y : I\!R \to I\!R$ is defined by

$$f_y(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{4}x^2 & \text{if } 0 < x \leq 2y \\ y(x - y) & \text{if } x > 2y. \end{cases}$$

In this paper the following properties of the functions $\{f_y\}_{y>0}$ play an important role.

Proposition 1 *The functions $\{f_y\}_{y>0}$ have the following properties:*

(i) f_y is non-decreasing on $I\!R$ for every $y > 0$;

(ii) f_y is convex on $I\!R$ for every $y > 0$, i.e., for every $x \in I\!R$, $a > 0$, and $c \geq 0$ we have

$$f_y(x) - f_y(x - a) \geq f_y(x - c) - f_y(x - c - a);$$

(iii) for every $y_1 > 0$, $y_2 > 0$ and $x \in I\!R$ we have $f_{y_1}(x) - f_{y_1}(x - y_2) \geq f_{y_2}(x - y_1) - f_{y_2}(x - 2y_1)$.

Proof

(i) Follows directly from the fact that f'_y is non-negative on $I\!R$, since

$$f'_y(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2}x & \text{if } 0 < x \leq 2y \\ y & \text{if } x > 2y. \end{cases}$$

(ii) Follows by the fact that f'_y is non-decreasing on $I\!R$.

(iii) First note that $f_{y_1}(x) + f_{y_2}(x - 2y_1) = f_{y_1+y_2}(x)$ for every $x \in I\!R$. Define the C^1-function $u : I\!R \to I\!R$ by

$$u(x) = f_{y_1+y_2}(x) - f_{y_1}(x - y_2) - f_{y_2}(x - y_1).$$

Clearly $u(x) = 0$ for $x \leq 0$ and $x \geq 2(y_1 + y_2)$. Moreover, for every $x \in (0, y_1 + y_2)$ we have

$$\begin{aligned} u'(x) &= \tfrac{1}{2}x - \max\{\tfrac{1}{2}(x - y_2), 0\} - \max\{\tfrac{1}{2}(x - y_1), 0\} \\ &= \tfrac{1}{2}x - \max\{\tfrac{1}{2}(2x - y_1 - y_2), \tfrac{1}{2}(x - y_2), \tfrac{1}{2}(x - y_1), 0\} \\ &> 0, \end{aligned}$$

and for every $x \in (y_1 + y_2, 2y_1 + 2y_2)$ we have

$$\begin{aligned} u'(x) &= \tfrac{1}{2}x - \min\{\tfrac{1}{2}(x - y_2), y_1\} - \min\{\tfrac{1}{2}(x - y_1), y_2\} \\ &= \tfrac{1}{2}x - \min\{\tfrac{1}{2}(2x - y_1 - y_2), \tfrac{1}{2}(x + y_2), \tfrac{1}{2}(x + y_1), y_1 + y_2\} \\ &< 0. \end{aligned}$$

So, u is increasing on $[0, y_1 + y_2]$ and decreasing on $[y_1 + y_2, 2(y_1 + y_2)]$. Therefore, $u(x) \geq 0$ for every $x \in [0, 2(y_1 + y_2)]$.

■

In this paper we also need the following minimax theorem.

Proposition 2 *Let X and Z be compact topological spaces and let $K : X \times Z \to I\!R$ be a continuous function. Suppose, moreover, that there exists a $z^* \in Z$ such that for every $x \in X$ we have $\min_{z \in Z} K(x, z) = K(x, z^*)$. Then we have*

$$\max_{x \in X} \min_{z \in Z} K(x, z) = \min_{z \in Z} \max_{x \in X} K(x, z) = \max_{x \in X} K(x, z^*).$$

Proof Obviously, we have

$$\begin{aligned}\inf_{z\in Z}\max_{x\in X} K(x,z) &\leq \max_{x\in X} K(x,z^*)\\ &= \max_{x\in X}\min_{z\in Z} K(x,z)\\ &\leq \inf_{z\in Z}\max_{x\in X} K(x,z),\end{aligned}$$

which finishes the proof. ■

We conclude this section with some formal definitions on cooperative games.

Definition 2 A *cooperative game* is a tuple (N, v), where N is the set of players and $v : 2^N \to I\!R$ its characteristic function, with the convention that $v(\emptyset) = 0$. The *core* of a cooperative game (N, v) is the set

$$C(v) = \{x \in I\!R^N \,:\, x(N) = v(N), x(S) \geq v(S) \text{ for every } S \subseteq N\}.$$

If $C(v) \neq \emptyset$ the game (N, v) is *balanced*. A cooperative game (N, v) is *totally balanced* if for every $S \subseteq N, S \neq \emptyset$ the subgame (S, v_S), where v_S is the restriction of v to 2^S, is balanced. A cooperative game (N, v) is *convex* (Shapley (1971)) if for every $i, j \in N$ and every $S \subseteq N\backslash\{i, j\}$ we have $v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\})$.

3 Oligopoly games

An *oligopoly with linear inverse demand function* is completely described by the tuple $(N, (y_i)_{i\in N}, (c_i)_{i\in N}, b)$. Here $N = \{1, \ldots, n\}$ is the collection of firms, $y_i > 0$ is the maximal production (capacity) of firm i, $c_i > 0$ is the marginal cost of firm i, i.e. the cost for the production of one unit of output. The constant $b \geq 0$ is the intercept of the inverse demand function: if total production is x then the price per unit of output is $\max\{b-x, 0\} = (b-x)_+$. Without loss of generality we assume that the firms are ranked according to their marginal costs, i.e. $c_1 \leq c_2 \leq \cdots \leq c_n$.
Corresponding to the oligopoly above the strategic *oligopoly game* $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ is defined by

$$X_i = [0, y_i] \tag{2}$$

for every $i \in N$, and

$$u_i(x) = (b - x(N))_+ \cdot x_i - c_i x_i \tag{3}$$

for every $i \in N$ and every $x = (x_i)_{i \in N} \in X_N$.
Aumann (1959) introduced two ways of converting a non-cooperative game $(N, (X_i)_{i \in N}, (u_i)_{i \in N})$ into a cooperative game. The first approach leads to the cooperative game (N, v_α) which is obtained by computing for every coalition S the amount of money which the players in S can guarantee themselves regardless what the players outside S do. The second approach results in the game (N, v_β) by computing for every coalition S the minimal amount of money which is such that the players outside S can prevent that the players in S get more. In the next subsection we will show that for oligopoly games these two cooperative games coincide, leading to the class of cooperative oligopoly games without transferable technologies. In subsection 3.2 the approach of Zhao (1999) is followed, where every member in a coalition can produce according to the cheapest technology present in this coalition. Again, in the spirit of Aumann (1959), two cooperative games are defined which turn out to coincide. This leads to the class of cooperative oligopoly games with transferable technologies.

3.1 Cooperative oligopoly games without transferable technologies

Let $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$ be an oligopoly with linear inverse demand function and let $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ be the corresponding oligopoly game as defined by (2) and (3). The cooperative game (N, v_α^{nt}) is defined by

$$\begin{aligned} v_\alpha^{nt}(S) &= \max_{x \in X_S} \min_{z \in X_{N \setminus S}} \sum_{i \in S} u_i(x, z) \\ &= \max_{x \in X_S} \min_{z \in X_{N \setminus S}} [(b - z(N \setminus S) - x(S))_+ \cdot x(S) - \sum_{i \in S} c_i x_i] \end{aligned} \tag{4}$$

for every $S \subset N$, $S \neq \emptyset$ and

$$\begin{aligned} v_\alpha^{nt}(N) &= \max_{x \in X_N} \sum_{i \in N} u_i(x) \\ &= \max_{x \in X_N} [(b - x(N))_+ \cdot x(N) - \sum_{i \in N} c_i x_i], \end{aligned} \tag{5}$$

whereas the cooperative game (N, v_β^{nt}) is defined by

$$\begin{aligned} v_\beta^{nt}(S) &= \min_{z \in X_{N \setminus S}} \max_{x \in X_S} \sum_{i \in S} u_i(x, z) \\ &= \min_{z \in X_{N \setminus S}} \max_{x \in X_S} [(b - z(N \setminus S) - x(S))_+ \cdot x(S) - \sum_{i \in S} c_i x_i] \end{aligned} \tag{6}$$

for every $S \subset N$, $S \neq \emptyset$ and

$$\begin{aligned} v_\beta^{nt}(N) &= \max_{x \in X_N} \sum_{i \in N} u_i(x) \\ &= [\max_{x \in X_N} (b - x(N))_+ \cdot x(N) - \sum_{i \in N} c_i x_i]. \end{aligned} \tag{7}$$

The superscript nt in (N, v_α^{nt}) and (N, v_β^{nt}) is an abbreviation for 'not transferable'. The following proposition shows that the games (N, v_α^{nt}) and (N, v_β^{nt}) coincide.

Proposition 3 *Let $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$ be an oligopoly with linear inverse demand function and let (N, v_α^{nt}) and (N, v_β^{nt}) be defined by (4)-(7). Then*

$$v_\alpha^{nt}(S) = v_\beta^{nt}(S) = \max_{x \in X_S} [(b - y(N \backslash S) - x(S)) \cdot x(S) - \sum_{i \in S} c_i x_i] \tag{8}$$

for every $S \subseteq N$.

Proof Let $S \subseteq N$ and define the continuous functions $f_S : X_S \to I\!R$ and $g_S : X_S \to I\!R$ by

$$f_S(x) = (b - y(N \backslash S) - x(S))_+ \cdot x(S) - \sum_{i \in S} c_i x_i,$$

and

$$g_S(x) = (b - y(N \backslash S) - x(S)) \cdot x(S) - \sum_{i \in S} c_i x_i$$

for every $x \in X_S$. Clearly, we have $f_S(x) \geq g_S(x)$ for every $x \in X_S$. If $\hat{x} \in X_S$ is such that $f_S(\hat{x}) > g_S(\hat{x})$ then $b - y(N \backslash S) - \hat{x}(S) < 0$ and $\hat{x}(S) > 0$. Hence $f_S(\hat{x}) = -\sum_{i \in S} c_i \hat{x}_i < 0 = f_S(x^0)$, where $x^0 \in X_S$ is given by $x_i^0 = 0$ for every $i \in S$, which implies $f_S(\hat{x}) < \max_{x \in X_S} f_S(x)$. As a consequence we get

$$\max_{x \in X_S} f_S(x) = \max_{x \in X_S} g_S(x).$$

Therefore, in order to establish (8), it is sufficient to prove that

$$v_\alpha^{nt}(S) = v_\beta^{nt}(S) = \max_{x \in X_S} f_S(x)$$

for every $S \subseteq N$. Note that

$$v_\alpha^{nt}(N) = v_\beta^{nt}(N) = \max_{x \in X_N} f_N(x)$$

by definition. Therefore, let $S \subset N$, $S \neq \emptyset$. Define the compact sets $X = X_S = \Pi_{i \in S}[0, y_i]$, $Z = X_{N \backslash S} = \Pi_{i \in N \backslash S}[0, y_i]$, and the continuous function $K : X \times Z \to I\!R$ by

$$K(x, z) = \sum_{i \in S} u_i(x, z) = (b - z(N \backslash S) - x(S))_+ \cdot x(S) - \sum_{i \in S} c_i x_i.$$

Define $z^* \in Z$ by $z_i^* = y_i$ for every $i \in N \backslash S$. One easily verifies that $\min_{z \in Z} K(x, z) = K(x, z^*) = f_S(x)$ for every $x \in X$. Application of Proposition 2 yields

$$v_\alpha^{nt}(S) = v_\beta^{nt}(S) = \max_{x \in X_S} f_S(x).$$

■

Definition 3 Let $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$ be an oligopoly with linear inverse demand function. The game (N, v_α^{nt}), defined by (4) and (5) (which is equal to the game (N, v_β^{nt}) defined by (6) and (7)), is called the *cooperative oligopoly game without transferable technologies* corresponding to the oligopoly $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$, and will be denoted by (N, v^{nt}).

The following proposition indicates how to compute the values of coalitions in cooperative oligopoly games without transferable technologies.

Proposition 4 *Let (N, v^{nt}) be the cooperative oligopoly game without transferable technologies, corresponding to the oligopoly $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$. Then, for every $S \subseteq N$, we have*

$$v^{nt}(S) = \sum_{j \in S} f_{y_j}(b - c_j - y(N \backslash S) - 2y_{S,j}), \tag{9}$$

where

$$y_{S,j} = \sum_{\substack{k \in S \\ k < j}} y_k.$$

Proof Let $S \subseteq N$, $S \neq \emptyset$. Write $S = \{s_1, \ldots, s_k\}$ with $s_1 < \cdots < s_k$, and let $b' = b - y(N \backslash S)$. According to Proposition 3 we have

$$v^{nt}(S) = \max_{x \in X_S} g_S(x), \tag{10}$$

where the C$^\infty$-function $g_S : X_S \to I\!R$ is given by

$$g_S(x) = (b' - x(S)) \cdot x(S) - \sum_{i \in S} c_i x_i.$$

Let $x^* = (x_i^*)_{i \in S} \in X_S$ be such that $v^{nt}(S) = g_S(x^*)$. In order to maximize g_S one should use the firms with lowest marginal cost first. For, if $i, j \in S$ with $i < j$ are such that $x_i^* < y_i$ and $x_j^* > 0$ then a decrease of x_j^* by some amount $\varepsilon > 0$ and an increase of x_i^* by the same amount does not decrease the value of the objective function g_S. Hence, we may assume that x^* is such that $x_j^* > 0$ for some $j \in S$ implies $x_i^* = y_i$ for every $i \in S$ with $i < j$. In order to prove (9) we will distinguish four cases.

Case 1: $x_{s_l}^* = 0$ for every $l \in \{1, \ldots, k\}$.

Since $x_{s_1}^* = 0$ and g_S is maximal at x^* we have

$$\frac{\partial g_S}{\partial x_{s_1}}(x^*) = b' - c_{s_1} \leq 0.$$

As a consequence, $b' - c_j - 2y_{S,j} \leq b' - c_{s_1} \leq 0$ for every $j \in S$. Therefore, $v^{nt}(S) = g_S(x^*) = 0 = \sum_{j \in S} f_{y_j}(b' - c_j - y_{S,j})$.

Case 2: there is an $m \in \{1, \ldots, k-1\}$ with $x_{s_l}^* = y_{s_l}$ for every $l \in \{1, \ldots, m\}$ and $x_{s_l}^* = 0$ for every $l \in \{m+1, \ldots, k\}$.

Since $x_{s_m}^* = y_{s_m}$, $x_{s_{m+1}}^* = 0$ and g_S is maximal at x^* we have

$$\frac{\partial g_S}{\partial x_{s_m}}(x^*) = b' - 2y_{S,s_{m+1}} - c_{s_m} \geq 0,$$

and

$$\frac{\partial g_S}{\partial x_{s_{m+1}}}(x^*) = b' - 2y_{S,s_{m+1}} - c_{s_{m+1}} \leq 0.$$

For every $j \in S, j \leq s_m$ we have $b' - c_j - 2y_{S,j} \geq 2y_{S,s_{m+1}} + c_{s_m} - c_j - 2y_{S,j} \geq 2y_j$, and hence $f_{y_j}(b' - c_j - 2y_{S,j}) = y_j(b' - c_j - 2y_{S,j} - y_j)$. For every $j \in S, j \geq s_{m+1}$ we have $b' - c_j - 2y_{S,j} \leq 2y_{S,s_{m+1}} + c_{s_{m+1}} - c_j - 2y_{S,j} \leq 0$, and hence $f_{y_j}(b' - c_j - 2y_{S,j}) = 0$. Therefore,

$$\begin{aligned} v^{nt}(S) = g_S(x^*) &= (b' - \sum_{l=1}^{m} y_{s_l}) \sum_{l=1}^{m} y_{s_l} - \sum_{l=1}^{m} c_{s_l} y_{s_l} \\ &= y_{s_1}(b' - c_{s_1} - y_{s_1}) + \\ &\quad y_{s_2}(b' - c_{s_2} - 2y_{s_1} - y_{s_2}) + \\ &\quad y_{s_3}(b' - c_{s_3} - 2y_{s_1} - 2y_{s_2} - y_{s_3}) + \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& +y_{s_m}(b' - c_{s_m} - 2\sum_{l=1}^{m-1} y_{s_l} - y_{s_m}) \\
= & \sum_{l=1}^{m} y_{s_l}(b' - c_{s_l} - 2y_{S,s_l} - y_{s_l}) \\
= & \sum_{j\in S} f_{y_j}(b' - c_j - 2y_{S,j}).
\end{aligned}$$

Case 3: there is an $m \in \{1,\ldots,k\}$ with $0 < x^*_{s_m} < y_{s_m}$ (and hence $x^*_{s_l} = y_{s_l}$ for every $l \in \{1,\ldots,m-1\}$ and $x^*_{s_l} = 0$ for every $l \in \{m+1,\ldots,k\}$).
Since g_S is maximal at x^* we have

$$\frac{\partial g_S}{\partial x_{s_m}}(x^*) = b' - 2\sum_{l=1}^{m} x^*_{s_l} - c_{s_m} = 0$$

or, equivalently, $x^*_{s_m} = \frac{1}{2}(b' - c_{s_m} - 2y_{S,s_m})$. Hence $b' - c_{s_m} - 2y_{S,s_m} = 2x^*_{s_m} \in (0, 2y_{s_m})$, and, as a consequence, $f_{y_{s_m}}(b' - c_{s_m} - 2y_{S,s_m}) = \frac{1}{4}(b' - c_{s_m} - 2y_{S,s_m})^2$. Moreover, for every $j \in S, j < s_m$ we have $b' - c_j - 2y_{S,j} \geq 2y_{S,s_m} + c_{s_m} - c_j - 2y_{S,j} \geq 2y_j$, and hence $f_{y_j}(b' - c_j - 2y_{S,j}) = y_j(b' - c_j - 2y_{S,j} - y_j)$. For every $j \in S, j \geq s_{m+1}$ we have $b' - c_j - 2y_{S,j} \leq 2y_{S,s_{m+1}} + c_{s_m} - c_j - 2y_{S,j} \leq 0$, and hence $f_{y_j}(b' - c_j - 2y_{S,j}) = 0$. Therefore,

$$\begin{aligned}
v^{nt}(S) = g_S(x^*) = & (b' - \sum_{l=1}^{m} x^*_{s_l})\sum_{l=1}^{m} x^*_{s_l} - \sum_{l=1}^{m} c_{s_l}x^*_{s_l} \\
= & (b' - \sum_{l=1}^{m-1} y_{s_l} - x^*_{s_m})(\sum_{l=1}^{m-1} y_{s_l} + x^*_{s_m}) \\
& -\sum_{l=1}^{m-1} c_{s_l}y_{s_l} - c_{s_m}x^*_{s_m} \\
= & (b' - \sum_{l=1}^{m-1} y_{s_l})(\sum_{l=1}^{m-1} y_{s_l}) - \sum_{l=1}^{m-1} c_{s_l}y_{s_l} \\
& +x^*_{s_m}(b' - c_{s_m} - 2y_{S,s_m} - x^*_{s_m}) \\
= & y_{s_1}(b' - c_{s_1} - y_{s_1}) + \\
& y_{s_2}(b' - c_{s_2} - 2y_{s_1} - y_{s_2}) + \\
& y_{s_3}(b' - c_{s_3} - 2y_{s_1} - 2y_{s_2} - y_{s_3}) + \\
& \vdots
\end{aligned}$$

$$
\begin{aligned}
&+y_{s_{m-1}}(b' - c_{s_{m-1}} - 2\sum_{l=1}^{m-2} y_{s_l} - y_{s_{m-1}})+\\
&\tfrac{1}{4}(b' - c_{s_m} - 2y_{S,s_m})^2\\
= \; & \sum_{j\in S} f_{y_j}(b' - c_j - 2y_{S,j}).
\end{aligned}
$$

Case 4: $x^*_{s_l} = y_{s_l}$ for every $l \in \{1, \ldots, k\}$.
Since $x^*_{s_k} = y_{s_k}$ and g_S is maximal at x^* we have

$$\frac{\partial g_S}{\partial x_{s_k}}(x^*) = b' - 2y(S) - c_{s_k} \geq 0.$$

For every $j \in S$ we have $b' - c_j - 2y_{S,j} \geq 2y(S) + c_{s_k} - c_j - 2y_{S,j} \geq 2y_j$ and hence $f_{y_j}(b' - c_j - 2y_{S,j}) = y_j(b' - c_j - 2y_{S,j} - y_j)$. Therefore,

$$
\begin{aligned}
v^{nt}(S) = g_S(x^*) &= (b' - \sum_{l=1}^{k} y_{s_l})\sum_{l=1}^{k} y_{s_l} - \sum_{l=1}^{k} c_{s_l} y_{s_l}\\
&= y_{s_1}(b' - c_{s_1} - y_{s_1})+\\
&\quad y_{s_2}(b' - c_{s_2} - 2y_{s_1} - y_{s_2})+\\
&\quad y_{s_3}(b' - c_{s_3} - 2y_{s_1} - 2y_{s_2} - y_{s_3})+\\
&\quad \vdots\\
&\quad +y_{s_k}(b' - c_{s_k} - 2\sum_{l=1}^{k-1} y_{s_l} - y_{s_k})\\
&= \sum_{l=1}^{k} y_{s_l}(b' - c_{s_l} - 2y_{S,s_l} - y_{s_l})\\
&= \sum_{j\in S} f_{y_j}(b' - c_j - 2y_{S,j}).
\end{aligned}
$$

■

Remark Proposition 4 states that in order to compute the value $v^{nt}(S)$ one has to solve a monopolistic optimization problem for every firm in S. If $S = \{s_1, s_2, \ldots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$, one first has to solve a monopolistic optimization problem for firm s_1, i.e. the firm in S with lowest marginal cost. Which monopolistic optimization problem (v, y) should be solved for firm s_1? Clearly, y should be equal to y_{s_1}, the maximum capacity of firm s_1. Furthermore, the number v should express approximately the profit per unit of output, if firm s_1 produces a very small amount of output. Taking into account that the player in $N\backslash S$ produce at maximal capacity in order to decrease the profits for S as much as possible and the fact that firm s_1 produces at costs c_{s_1} per unit of output, we get $v = b - y(N\backslash S) - c_{s_1}$. Which monopolistic optimization problem (v, y) should be solved for firm s_2? Of course y should be equal to y_{s_2}. If coalition S is going to use the capacity of firm s_2, it is clear that S uses firm s_1 already at full capacity. So, it is tempting to conclude that $v = b - y(N\backslash S) - c_{s_2} - y_{s_1}$. However, by doing this one neglects the fact that every unit produced by firm s_2 decreases the selling price by one unit, and, as a consequence, the profits generated by firm s_1 by y_{s_1} units. In order to take also this effect into account one should use $v = b - y(N\backslash S) - c_{s_2} - 2y_{s_1}$. Similar arguments can be used for firms $s_3, \ldots, s_k$.

Example 1 Consider the cooperative oligopoly game without transferable technologies (N, v^{nt}), corresponding to the oligopoly $(N, (y_i)_{i\in N}, (c_i)_{i\in N}, b)$ with $N = \{1, 2, 3\}$, $y_1 = 14$, $y_2 = 8$, $y_3 = 12$, $c_1 = 2$, $c_2 = 4$, $c_3 = 16$, and $b = 60$. How to calculate e.g. $v^{nt}(\{1, 3\})$? Firm 1 faces the monopolistic optimization problem (x, y) with $x = b - y_2 - c_1 = 50$ and $y = y_1 = 14$, so its profit is $f_y(x) = f_{14}(50) = 504$. Firm 3 faces the monopolistic optimization problem (x, y) with $x = b - y_2 - c_3 - 2y_1 = 8$ and $y = y_3 = 12$, so its profit is $f_y(x) = f_{12}(8) = 16$. Therefore, $v^{nt}(\{1, 3\}) = 504 + 16 = 520$. Proceeding in this way we get the following table

S	contribution of player 1	contribution of player 2	contribution of player 3	$v^{nt}(S)$
$\{1\}$	$f_{14}(38) = 336$	–	–	336
$\{2\}$	–	$f_8(30) = 176$	–	176
$\{3\}$	–	–	$f_{12}(22) = 121$	121
$\{1, 2\}$	$f_{14}(46) = 448$	$f_8(16) = 64$	–	512
$\{1, 3\}$	$f_{14}(50) = 504$	–	$f_{12}(8) = 16$	520
$\{2, 3\}$	–	$f_8(42) = 272$	$f_{12}(14) = 49$	321
$\{1, 2, 3\}$	$f_{14}(58) = 616$	$f_8(28) = 160$	$f_{12}(0) = 0$	776

The last column of this table provides the game (N, v^{nt}).

3.2 Cooperative oligopoly games with transferable technologies

In this subsection we follow the approach of Zhao (1999) where cooperating firms can use the cheapest technology available. Consider, once again, the oligopoly $(N, (y_i)_{i\in N}, (c_i)_{i\in N}, b)$. In the spirit of Aumann (1959), the cooperative game (N, v^t_α) is defined by

$$v^t_\alpha(S) = \max_{x \in X_S} \min_{z \in X_{N \setminus S}} [(b - z(N \setminus S) - x(S))_+ \cdot x(S) - \underline{c}_S \cdot x(S)] \tag{11}$$

for every $S \subset N$, $S \neq \emptyset$ and

$$v^t_\alpha(N) = \max_{x \in X_N} [(b - x(N))_+ \cdot x(N) - \underline{c}_N \cdot x(N)], \tag{12}$$

whereas the cooperative game (N, v^t_β) is defined by

$$v^t_\beta(S) = \min_{z \in X_{N \setminus S}} \max_{x \in X_S} [(b - z(N \setminus S) - x(S))_+ \cdot x(S) - \underline{c}_S \cdot x(S)] \tag{13}$$

for every $S \subset N$, $S \neq \emptyset$ and

$$v^t_\beta(N) = \max_{x \in X_N} [(b - x(N))_+ \cdot x(N) - \underline{c}_N \cdot x(N)]. \tag{14}$$

The superscript t in (N, v^t_α) and (N, v^t_β) is an abbreviation for 'transferable'. Once again it can be shown that the games (N, v^t_α) and (N, v^t_β) coincide (see also Zhao (1999)).

Proposition 5 *Let $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$ be an oligopoly with linear inverse demand function and let (N, v^t_α) and (N, v^t_β) be defined by (11)-(14). Then*

$$v^{nt}_\alpha(S) = v^{nt}_\beta(S) = \max_{x \in X_S} [(b - y(N \setminus S) - x(S)) \cdot x(S) - \underline{c}_S \cdot x(S)] \tag{15}$$

for every $S \subseteq N$.

Proof The proof is analogous to the proof of Proposition 3 with functions functions $f_S : X_S \to I\!R$ and $g_S : X_S \to I\!R$ defined by

$$f_S(x) = (b - y(N \setminus S) - x(S))_+ \cdot x(S) - \underline{c}_S \cdot x(S),$$

and

$$g_S(x) = (b - y(N \setminus S) - x(S)) \cdot x(S) - \underline{c}_S \cdot x(S)$$

for every $x \in X_S$. ■

Definition 4 Let $(N,(y_i)_{i\in N},(c_i)_{i\in N},b)$ be an oligopoly with linear inverse demand function. The game (N,v^t_α), defined by (11) and (12) (which is equal to the game (N,v^t_β) defined by (13) and (14)), is called the *cooperative oligopoly game with transferable technologies* corresponding to the oligopoly $(N,(y_i)_{i\in N},(c_i)_{i\in N},b)$, and will be denoted by (N,v^t).

The following proposition shows how to compute the values of coalitions in cooperative oligopoly games with transferable technologies.

Proposition 6 *Let (N,v^t) be the cooperative oligopoly game with transferable technologies, corresponding to the oligopoly $(N,(y_i)_{i\in N},(c_i)_{i\in N},b)$. Then, for every $S\subseteq N$, we have*

$$v^t(S)=\sum_{j\in S} f_{y_j}(b-\underline{c}_S-y(N\backslash S)-2y_{S,j}). \tag{16}$$

Proof Let $S\subseteq N$, $S\neq\emptyset$ and let $b'=b-y(N\backslash S)$. According to (15) we have

$$\begin{aligned} v^t(S) &= \max_{x\in X_S}[(b'-x(S))x(S)-\underline{c}_S\cdot x(S)] \\ &= \max_{t\in[0,y(S)]} g_S(t), \end{aligned}$$

where the C^∞-function $g_S:[0,y(S)]\rightarrow I\!R$ is given by $g_S(t)=(b'-\underline{c}_S-t)t$ for every $t\in[0,y(S)]$. Let $t^*\in[0,y(S)]$ be such that $v^t(S)=g_S(t^*)$. In order to prove (16) we distinguish three cases.
<u>Case 1:</u> $t^*=0$.
Since g_S is maximal at $t^*=0$ we have $g'_S(0)=b'-\underline{c}_S\leq 0$. Consequently we have $b'-\underline{c}_S-2y_{S,j}\leq 0$ for every $j\in S$ and hence $v^t(S)=g_S(0)=0=\sum_{j\in S} f_{y_j}(b'-\underline{c}_S-2y_{S,j})$.
<u>Case 2:</u> $t^*\in(0,y(S))$.
Since g_S is maximal at t^* we have $g'_S(t^*)=b'-\underline{c}_S-2t^*=0$, and hence $t^*=\frac{1}{2}(b'-\underline{c}_S)$. Let $j^*\in S$ be such that $t^*\in(y_{S,j^*},y_{S,j^*}+y_{j^*}]$. Then $b'=\underline{c}_S+2t^*\in(\underline{c}_S+2y_{S,j^*},\underline{c}_S+2y_{S,j^*}+2y_{j^*}]$, and hence $f_{y_{j^*}}(b'-\underline{c}_S-2y_{S,j^*})=\frac{1}{4}(b'-\underline{c}_S-2y_{S,j^*})^2$. Moreover, for every $j\in S, j<j^*$ we have $b'-\underline{c}_S-2y_{S,j}\geq 2y_{S,j^*}-2y_{S,j}\geq 2y_j$, and hence $f_{y_j}(b'-\underline{c}_S-2y_{S,j})=y_j(b'-\underline{c}_S-2y_{S,j}-y_j)$. For every $j\in S, j>j^*$ we have $b'-\underline{c}_S-2y_{S,j}\leq 2y_{S,j^*}+2y_{j^*}-2y_{S,j}\leq 0$, and hence $f_{y_j}(b'-\underline{c}_S-2y_{S,j})=0$. Therefore,

$$\begin{aligned} v^t(S)=g_S(t^*) &= (b'-\underline{c}_S-t^*)t^* \\ &= \tfrac{1}{4}(b'-\underline{c}_S)^2 \\ &= \sum_{j\in S,j<j^*} y_j(b'-\underline{c}_S-2y_{S,j}-y_j)+\tfrac{1}{4}(b'-\underline{c}_S-2y_{S,j^*})^2 \end{aligned}$$

$$= \sum_{j \in S} f_{y_j}(b' - \underline{c}_S - 2y_{S,j}).$$

Case 3: $t^* = y(S)$.
Since g_S is maximal at $t^* = y(S)$ we have $g'_S(t^*) = b' - \underline{c}_S - 2y(S) \geq 0$. For every $j \in S$ we have $b' - \underline{c}_S - 2y_{S,j} \geq 2y(S) - 2y_{S,j} \geq 2y_j$ and hence $f_{y_j}(b' - \underline{c}_S - 2y_{S,j}) = y_j(b' - \underline{c}_S - 2y_{S,j} - y_j)$. Therefore,

$$\begin{aligned} v^t(S) = g_S(t^*) &= (b' - \underline{c}_S - y(S))y(S) \\ &= \sum_{j \in S} y_j(b' - \underline{c}_S - 2y_{S,j} - y_j) \\ &= \sum_{j \in S} f_{y_j}(b' - \underline{c}_S - 2y_{S,j}). \end{aligned}$$

■

Remark Proposition 6 states that in order to compute the value $v^t(S)$ one has to solve a monopolistic optimization problem for every firm in S. The description of these monopolistic optimization problems is completely analogous to the description of the monopolistic optimization problems in the remark after Proposition 4. The only difference is that the marginal cost for every firm has to be replaced with the lowest marginal cost in the coalition.

Example 2 Consider the cooperative oligopoly game with transferable technologies (N, v^t), corresponding to the oligopoly $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$ in Example 1. How to calculate e.g. $v^t(\{1,2\})$? Firm 1 faces the monopolistic optimization problem (x, y) with $x = b - y_3 - c_1 = 46$ and $y = y_1 = 14$, so its profit is $f_y(x) = f_{14}(46) = 448$. Firm 2 faces the monopolistic optimization problem (x, y) with $x = b - y_3 - c_1 - 2y_1 = 18$ and $y = y_2 = 8$, so its profit is $f_y(x) = f_8(18) = 80$. Therefore, $v^t(\{1,2\}) = 448 + 80 = 528$. Proceeding in this way we get the following table

S	contribution of player 1	contribution of player 2	contribution of player 3	$v^t(S)$
$\{1\}$	$f_{14}(38) = 336$	–	–	336
$\{2\}$	–	$f_8(30) = 176$	–	176
$\{3\}$	–	–	$f_{12}(22) = 121$	121
$\{1,2\}$	$f_{14}(46) = 448$	$f_8(18) = 80$	–	528
$\{1,3\}$	$f_{14}(50) = 504$	–	$f_{12}(22) = 121$	625
$\{2,3\}$	–	$f_8(42) = 272$	$f_{12}(26) = 168$	440
$\{1,2,3\}$	$f_{14}(58) = 616$	$f_8(30) = 176$	$f_{12}(14) = 49$	841

The last column of this table provides the game (N, v^t).

4 Properties of oligopoly games

In this section we use the results of the previous section in order to prove some properties of cooperative oligopoly games. In subsection 4.1 we show that oligopoly games without transferable technologies are convex games and in subsection 4.2 we show that oligopoly games with transferable technologies are totally balanced.

4.1 Properties of oligopoly games without transferable technologies

In order to prove the main result of this section we first need a lemma.

Lemma 1 *Let (N, v^{nt}) be the cooperative oligopoly game without transferable technologies corresponding to the oligopoly $(N, (y_i)_{i\in N}, (c_i)_{i\in N}, b)$. Let $l, m \in N, l < m$ and let $S \subseteq N\backslash\{l, m\}$ be such that $S \cap \{l+1, ..., m-1\} = \emptyset$. Then we have*

$$v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S) \geq 0.$$

Proof We write $x_{T;i} = f_{y_i}(b - c_i - y(N\backslash T) - 2y_{T,i})$ for every $T \subseteq N$, $i \in T$. First note that

$$\begin{aligned}
&(x_{S\cup\{l,m\};l} - x_{S\cup\{l\};l}) + (x_{S\cup\{l,m\};m} - x_{S\cup\{m\};m}) \\
&= f_{y_l}(b - c_l - y(N\backslash(S \cup \{l, m\})) - 2y_{S\cup\{l,m\},l}) \\
&\quad - f_{y_l}(b - c_l - y(N\backslash(S \cup \{l\})) - 2y_{S\cup\{l\},l}) \\
&\quad + f_{y_m}(b - c_m - y(N\backslash(S \cup \{l, m\})) - 2y_{S\cup\{l,m\},m}) \\
&\quad - f_{y_m}(b - c_m - y(N\backslash(S \cup \{m\})) - 2y_{S\cup\{m\},m}) \\
&= f_{y_l}(b - c_l - y(N\backslash S) + y_l + y_m - 2y_{S,l}) \\
&\quad - f_{y_l}(b - c_l - y(N\backslash S) + y_l - 2y_{S,l}) \\
&\quad + f_{y_m}(b - c_m - y(N\backslash S) - y_l + y_m - 2y_{S,l}) \\
&\quad - f_{y_m}(b - c_m - y(N\backslash S) + y_m - 2y_{S,l}) \\
&\geq f_{y_l}(b - c_m - y(N\backslash S) + y_l + y_m - 2y_{S,l}) \\
&\quad - f_{y_l}(b - c_m - y(N\backslash S) + y_l - 2y_{S,l}) \\
&\quad + f_{y_m}(b - c_m - y(N\backslash S) - y_l + y_m - 2y_{S,l}) \\
&\quad - f_{y_m}(b - c_m - y(N\backslash S) + y_m - 2y_{S,l}) \\
&\geq 0,
\end{aligned}$$

where at the first inequality we used Proposition 1 (ii) with $y = y_l$, $x = b - c_l - y(N\backslash S) + y_l + y_m - 2y_{S,l}$, $a = y_m$, and $c = c_m - c_l$, and at the second inequality we used Proposition 1 (iii) with $x = b - c_m - y(N\backslash S) + y_l + y_m - 2y_{S,l}$, $y_1 = y_l$, and $y_2 = y_m$. Further, for $j \in S$, $j < l$ we have

$$\begin{aligned}
&x_{S\cup\{l,m\};j} - x_{S\cup\{l\};j} - x_{S\cup\{m\};j} + x_{S;j} \\
&= f_{y_j}(b - c_j - y(N\backslash(S\cup\{l,m\})) - 2y_{S\cup\{l,m\},j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash(S\cup\{l\})) - 2y_{S\cup\{l\},j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash(S\cup\{m\})) - 2y_{S\cup\{m\},j}) \\
&\quad + f_{y_j}(b - c_j - y(N\backslash S) - 2y_{S,j}) \\
&= f_{y_j}(b - c_j - y(N\backslash S) + y_l + y_m - 2y_{S,j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash S) + y_l - 2y_{S,j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash S) + y_m - 2y_{S,j}) \\
&\quad + f_{y_j}(b - c_j - y(N\backslash S) - 2y_{S,j}) \\
&\geq 0,
\end{aligned}$$

because of Proposition 1 (ii) with $y = y_j$, $x = b - c_j - y(N\backslash S) + y_l + y_m - 2y_{S,j}$, $a = y_m$, and $c = y_l$. Finally, for $j \in S$, $j > m$ we have

$$\begin{aligned}
&x_{S\cup\{l,m\};j} - x_{S\cup\{l\};j} - x_{S\cup\{m\};j} + x_{S;j} \\
&= f_{y_j}(b - c_j - y(N\backslash(S\cup\{l,m\})) - 2y_{S\cup\{l,m\},j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash(S\cup\{l\})) - 2y_{S\cup\{l\},j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash(S\cup\{m\})) - 2y_{S\cup\{m\},j}) \\
&\quad + f_{y_j}(b - c_j - y(N\backslash S) - 2y_{S,j}) \\
&= f_{y_j}(b - c_j - y(N\backslash S) - y_l - y_m - 2y_{S,j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash S) - y_l - 2y_{S,j}) \\
&\quad - f_{y_j}(b - c_j - y(N\backslash S) - y_m - 2y_{S,j}) \\
&\quad + f_{y_j}(b - c_j - y(N\backslash S) - 2y_{S,j}) \\
&\geq 0,
\end{aligned}$$

because of Proposition 1 (ii) with $y = y_j$, $x = b - c_j - y(N\backslash S) - 2y_{S,j}$, $a = y_l$, and $c = y_m$. According to Proposition 4 we have

$$\begin{aligned}
&v^{nt}(S\cup\{l,m\}) - v^{nt}(S\cup\{l\}) - v^{nt}(S\cup\{m\}) + v^{nt}(S) \\
&= \sum_{j\in S\cup\{l,m\}} x_{S\cup\{l,m\};j} - \sum_{j\in S\cup\{l\}} x_{S\cup\{l\};j} - \sum_{j\in S\cup\{m\}} x_{S\cup\{m\};j} + \sum_{j\in S} x_{S;j} \\
&= \sum_{j\in S} (x_{S\cup\{l,m\};j} - x_{S\cup\{l\};j} - x_{S\cup\{m\};j} + x_{S;j}) \\
&\quad + (x_{S\cup\{l,m\};l} - x_{S\cup\{l\};l}) + (x_{S\cup\{l,m\};m} - x_{S\cup\{m\};m}) \\
&\geq 0.
\end{aligned}$$

■

Theorem 1 *Every cooperative oligopoly game without transferable technologies is convex.*

Proof In order to show that every cooperative oligopoly game without transferable technologies is convex it suffices to show that for every cooperative oligopoly game without transferable technologies (N, v^{nt}), every $l, m \in N$ with $l < m$, and every $S \subseteq N \backslash \{l, m\}$ we have $v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S) \geq 0$. We will prove this with induction to $k = |S \cap \{l+1, \ldots, m-1\}|$, i.e. with induction to the number of players in S between l and m.

Induction basis: Let (N, v^{nt}) be a cooperative oligopoly game without transferable technologies, $l, m \in N$ with $l < m$, and $S \subseteq N \backslash \{l, m\}$ such that

$$|S \cap \{l+1, \ldots, m-1\}| = 0.$$

According to Lemma 1 we have $v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S) \geq 0$.

Induction step: Let $k \in \mathbb{N}$ and suppose that for every cooperative oligopoly game without transferable technologies (N, v^{nt}), every $l, m \in N$ with $l < m$, every $S \subseteq N \backslash \{l, m\}$ such that

$$|S \cap \{l+1, \ldots, m-1\}| \leq k-1$$

we have $v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S) \geq 0$.

Let (N, v^{nt}) be a cooperative oligopoly game without transferable technologies, corresponding to the oligopoly $(N, (y_i)_{i \in N}, (c_i)_{i \in N}, b)$, let $l, m \in N$ with $l < m$, and let $S \subseteq N \backslash \{l, m\}$ be such that

$$|S \cap \{l+1, \ldots, m-1\}| = k.$$

Define $l^* = \min S \cap \{l+1, \ldots, m-1\}$. Let (N, w^{nt}) be the cooperative oligopoly game without transferable technologies corresponding to the oligopoly $(N, (y_i')_{i \in N}, (c_i')_{i \in N}, b)$, defined by $y_i' = y_i$ for every $i \in N$, $c_i' = c_{l^*}$ for every $i \in \{l, l+1, \ldots, l^*\}$, $c_i' = c_i$ for all other i. Then, we find

$$\begin{aligned}
&(v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S)) \\
&\quad -(w^{nt}(S \cup \{l, m\}) - w^{nt}(S \cup \{l\}) - w^{nt}(S \cup \{m\}) + w^{nt}(S)) \\
&= f_{y_l}(b - c_l - y(N \backslash S) + y_l + y_m - 2y_{S,l}) \\
&\quad -f_{y_l}(b - c_l - y(N \backslash S) + y_l - 2y_{S,l}) \\
&\quad -f_{y_l}(b - c_{l^*} - y(N \backslash S) + y_l + y_m - 2y_{S,l}) \\
&\quad +f_{y_l}(b - c_{l^*} - y(N \backslash S) + y_l - 2y_{S,l}) \\
&\geq 0,
\end{aligned} \tag{17}$$

where the equality follows by using (9) and the fact that player l is the only player in $S \cup \{l, m\}$ whose marginal cost has been changed. The inequality follows by Proposition 1 (ii) with $y = y_l$, $x = b - c_l - y(N \backslash S) + y_l + y_m - 2y_{S,l}$, $a = y_m$, and $c = c_{l^*} - c_l$.

Let (N, u^{nt}) be the cooperative oligopoly game without transferable technologies, corresponding to the oligopoly $(N, (y_i'')_{i \in N}, (c_i'')_{i \in N}, b)$, defined by $y_{l^*}'' = y_l'$, $y_l'' = y_{l^*}'$, $y_i'' = y_i'$ for every $i \in N \backslash \{l, l^*\}$, and $c_i'' = c_i'$ for every $i \in N$. Define $S' = (S \cup \{l\}) \backslash \{l^*\}$. One easily verifies that $u^{nt}(S' \cup \{l^*, m\}) = w^{nt}(S \cup \{l, m\})$, $u^{nt}(S' \cup \{l^*\}) = w^{nt}(S \cup \{l\})$, $u^{nt}(S' \cup \{m\}) = w^{nt}(S \cup \{m\})$, and $u^{nt}(S') = w^{nt}(S)$. Since $|S' \cap \{l^* + 1, \ldots, m - 1\}| = k - 1$ we get by induction hypothesis

$$u^{nt}(S' \cup \{l^*, m\}) - u^{nt}(S' \cup \{l^*\}) - u^{nt}(S' \cup \{m\}) + u^{nt}(S') \geq 0,$$

and hence

$$w^{nt}(S \cup \{l, m\}) - w^{nt}(S \cup \{l\}) - w^{nt}(S \cup \{m\}) + w^{nt}(S) \geq 0.$$

By (17) we infer that

$$v^{nt}(S \cup \{l, m\}) - v^{nt}(S \cup \{l\}) - v^{nt}(S \cup \{m\}) + v^{nt}(S) \geq 0,$$

which finishes the proof. ∎

4.2 Properties of oligopoly games with transferable technologies

In subsection 4.1 we have seen that every cooperative oligopoly game without transferable technologies is convex. Zhao (1999) noted that this is not true in general for cooperative oligopoly games with transferable technologies. The next example illustrates this once again.

Example 3 Consider the cooperative oligopoly game with transferable technologies (N, v^t) of Example 2. One easily verifies that $v^t(123) - v^t(13) = 216 < 319 = v^t(23) - v^t(3)$, which shows that (N, v^t) is not convex.

However, we can show that every cooperative oligopoly game with transferable technologies is totally balanced. Moreover we can show that the marginal vector, corresponding to the order which ranks the firms in the order of increasing marginal costs, provides a core element.

Proposition 7 *Let (N, v^t) be a cooperative oligopoly game with transferable technologies, corresponding to the oligopoly $(N, (y_i)_{i\in N}, (c_i)_{i\in N}, b)$. Then (N, v^t) is totally balanced. Moreover, for the vector $x \in I\!R^N$, defined by*

$$x_i = v^t(\{j \in N : j \leq i\}) - v^t(\{j \in N : j \leq i-1\})$$

for every $i \in N$, we have $x \in C(v^t)$.

Proof First we note that every subgame (S, v^t_S) of (N, v^t), i.e. $v^t_S(T) = v^t(T)$ for every $T \subseteq S$, is a cooperative oligopoly game with transferable technologies itself. In order to see this let $S \subseteq N$, $S \neq \emptyset$. From (16) one easily derives that (S, v^t_S) is the cooperative oligopoly game with transferable technologies, corresponding to the oligopoly $(S, (c_i)_{i\in S}, (y_i)_{i\in S}, b')$, where $b' = (b - y(N\backslash S))_+$. Therefore, in order to prove the proposition, it is sufficient to show that (N, v^t) is balanced.
Therefore, let (N, w) be the cooperative oligopoly game with transferable technologies, corresponding to the oligopoly $(N, (y_i)_{i\in N}, (c'_i)_{i\in N}, b)$, where $c'_i = c_1$ for every $i \in N$. Since all firms have equal cost, the cooperative oligopoly games with and without transferable technologies, corresponding to this oligopoly, coincide. Hence, according to Theorem 1, (N, w) is convex. Define the marginal vector $y \in I\!R^N$ by

$$y_i = w(\{j \in N : j \leq i\}) - w(\{j \in N : j \leq i-1\})$$

for every $i \in N$. By convexity of (N, w) we have $y \in C(w)$. Moreover, due to (16), we have $v^t(S) = w(S)$ for every $S \subseteq N$ with $1 \in S$ and $v^t(S) \leq w(S)$ for all other S. As a consequence we get

$$x = y \in C(w) \subseteq C(v^t).$$

■

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